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Unraveling Financial Interconnections: A Methodical Investigation into the Application of Copula Theory in Modeling Asset Dependence

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Abstract: Copula theory, a branch of statistics and probability theory, focuses on characterizing and modeling the dependency structures between random variables. Within finance, copulas offer a versatile framework crucial for tasks like risk management, portfolio optimization, and derivative pricing. Despite its importance, applying copula theory in finance can be challenging due to its complexity and the unique features of financial time-series data. This method review explores the utilization of copula theory in modeling dependency among financial assets. It examines copula theory fundamentals, various modeling techniques, empirical applications in finance, future directions, and practical implementation. By synthesizing existing literature, this review aims to shed light on the strengths, limitations, and practical considerations of copula-based modeling within the finance domain.

Keywords: copula theory; stochastic model; dependence modeling; financial assets; method application

1. Introduction

In the intricate landscape of modern finance, characterized by intricate interdependencies among various financial assets, the accurate representation and modeling of dependency structures among these assets are essential for informed decision-making and effective risk mitigation strategies (Yu et al., 2024b). Amidst this backdrop, copula theory has emerged as a potent mathematical framework, offering a versatile toolset for modeling the complex interrelationships inherent in financial markets (Neumeyer et al., 2019; Segnon et al., 2024; Xiao et al., 2023; Yao & Li, 2023).

Originally conceived within the realm of statistics and probability theory, copula theory has found extensive applications across diverse fields due to its ability to capture and characterize the dependence structures among random variables independently of their marginal distributions (Czado, 2019; Nelsen, 2006). In finance, where the dynamics of asset returns and market interactions play a pivotal role in shaping investment strategies and risk management practices, the integration of copula theory presents a compelling opportunity for enhancing analytical methodologies and decision-making processes. The application of copula theory in finance holds significant potential for addressing the inherent challenges associated with modeling dependency among financial assets. Unlike traditional correlation-based approaches, which often overlook non-linear dependencies and fail to capture tail dependence, copulas offer a more nuanced and comprehensive framework for modeling joint distributions (Koopman et al., 2016; Yu et al., 2024b, 2024a). By explicitly modeling the dependency structure separate from the marginal distributions, copula-based models can better capture the intricate relationships observed in financial time-series data, thus providing more accurate insights into risk exposure and portfolio dynamics (Bedoui et al., 2023).

Despite its potential contributions, the integration of copula theory within financial modeling is not without challenges. Financial data often exhibit non-normality, time-varying volatility, and fat-tailed distributions, posing significant hurdles to traditional modeling techniques. Moreover, the complexity of copula theory itself, coupled with the need for careful calibration and validation, underscores the importance of a rigorous and systematic approach to its application in financial contexts.

This method review seeks to address these challenges by providing a comprehensive examination of the utilization of copula theory in modeling dependency among financial assets. By synthesizing existing literature, theoretical foundations, and empirical applications, this review aims to contribute to the ongoing discourse on the role of copula-based modeling in finance. By shedding light on the strengths, limitations, and practical considerations of copula-based modeling within the finance domain, this review aims to provide a valuable resource for practitioners and researchers seeking to leverage copula theory effectively in addressing the complex challenges of modern finance. Ultimately, it is hoped that this review will contribute to the advancement of analytical methodologies and decision-making frameworks in finance, thus enabling more robust and informed strategies for managing financial risk and optimizing investment portfolios.

2. Data and Method

2.1. Data Collection

Relevant literature in the Web of Science (WoS) core database was searched on 5 May 2024. To analyze the applications of the copula in finance, initially, we used the keywords "copula + finance," spanning all periods, with document type set to "article" and language set to "English." This search yielded a total of 1020 relevant articles from the literature. Subsequently, we utilized additional keywords including "copula + risk management," "copula + portfolio optimization," and "copula + derivative pricing" to extract further relevant literature for conducting keyword co-occurrence analysis.

2.2. Research Method

VOSviewer, developed by Eck & Waltman (2011), is a versatile software tool for visualizing and analyzing bibliometric networks. Widely used in academia and research, it empowers users to explore scientific landscapes, uncover trends, and map relationships between scholarly articles, authors, and keywords (Almeida & Gonçalves, 2023; Migliavacca et al., 2022; Shome et al., 2023). With its intuitive interface and robust analytical capabilities, VOSviewer enables researchers to reveal hidden patterns within large datasets, aiding in knowledge discovery across diverse fields. Its features include network visualization, clustering, density visualization, and customization options, making it an indispensable tool for bibliometric analysis, facilitating insights into scientific structures, trends, and potential collaborations.

3. Copula Theory Fundamentals

3.1. Definition of Copula

A d-dimensional copula C is a multivariate distribution function on the d-dimensional hypercube $[0,1]^d$ with uniformly distributed marginals. The corresponding copula density for an absolutely continuous copula we denote by c can be obtained by partial differentiation, i.e., $c(u_1,\dots,u_d) := \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C(u_1,\cdots,u_d)$ for all u in $[0,1]^d$. Sklar (1959) proved the following fundamental representation theorem for multivariate distributions in terms of their marginal distributions and a corresponding copula. Let x be a d-dimensional random vector with joint distribution function $F(x)$ and marginal distribution functions $F_i, i = 1,\dots,d$, then the joint distribution function can be expressed as

$$F(x_1,\dots,x_d) = C\big(F_1(x_1),\dots,F_d(x_d)\big)$$

with associated density or probability mass function

$$f(x_1,\dots,x_d) = c\big(F_1(x_1),\dots,F_d(x_d)\big) f_1(x_1) \cdots f_d(x_d)$$

for some d-dimensional copula C with copula density c. For absolutely continuous distributions, the copula C is unique. The inverse also holds: the copula corresponding to a multivariate distribution function $F(x)$ with marginal distribution functions F_i for $i = 1, \ldots, d$ can be expressed as

$$c(u_1, \ldots, u_d) = F\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right) = \frac{f\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right)}{f_1\left(F_1^{-1}(u_1)\right) \cdots f_d\left(F_d^{-1}(u_d)\right)}$$

3.2. Types of Copulas

There are three principal methodologies for constructing copulas. The initial approach entails the application of the probability integral transform to individual margins of established multivariate distributions, thereby engendering elliptical copulas (Nelsen et al., 2001). Noteworthy members of this class include the Gaussian and t-copulas. The second method involves the utilization of generator functions to deduce Archimedean copulas, incorporating well-known instances such as the Clayton, Gumbel, Frank, and Joe copulas (Cossette et al., 2019). Lastly, the third approach extends the univariate extreme-value theory to higher dimensions, resulting in the emergence of an additional category of copulas (Czado, 2019; Czado et al., 2013; Nelsen, 2006).

Table 1 outlines the various copula families. Symmetric and asymmetric copulas are fundamental in copula theory. Symmetric copulas remain invariant when the order of the variables is changed, making them suitable for situations where dependencies are uniform in all directions. In contrast, asymmetric copulas allow for different dependency structures in different directions, making them ideal for modeling scenarios with varying tail dependencies (Arbel et al., 2019; Xiao et al., 2023). For instance, the Clayton copula highlights lower tail dependence (Bevilacqua et al., 2024), whereas the Gumbel copula emphasizes upper tail dependence (Rašiová & Árendáš, 2023).

Table 1. Copula family set.

Copula family	***par***	***par2***
Gaussian	(−1, 1)	-
Student-t	(−1, 1)	(2, ∞)
(Survival) Clayton	(0, ∞)	-
Rotated Clayton (90 and 270 degrees)	(−∞, 0)	-
(Survival) Gumbel	[1, ∞)	-
Rotated Gumbel (90 and 270 degrees)	(−∞, −1]	-
Frank	R\{0}	-
(Survival) Joe	(1, ∞)	-
Rotated Joe (90 and 270 degrees)	(−∞, −1)	-
(Survival) Clayton-Gumbel (BB1)	(0, ∞)	[1, ∞)
Rotated Clayton-Gumbel (90 and 270 degrees)	(−∞, 0)	(−∞, −1]
(Survival) Joe-Gumbel (BB6)	[1, ∞)	[1, ∞)
Rotated Joe-Gumbel (90 and 270 degrees)	(−∞, −1]	(−∞, −1]
(Survival) Joe-Clayton (BB7)	[1, ∞)	(0, ∞)
Rotated Joe-Clayton (90 and 270 degrees)	(−∞, −1]	(−∞, 0)
(Survival) Joe-Frank (BB8)	[1, ∞)	(0, 1]
Rotated Joe-Frank (90 and 270 degrees)	(−∞, −1]	[−1, 0)
(Survival) Tawn type 1	[1, ∞)	[0, 1]
Rotated Tawn type 1(90 and 270 degrees)	(−∞, −1]	[0, 1]
(Survival) Tawn type 2	[1, ∞)	[0, 1]
Rotated Tawn type 2 (90 and 270 degrees)	(−∞, −1]	[0, 1]

Note: *par* denotes the first parameter of the copula function and *par2* represents the second parameter if applicable.

3.3. Marginal Distribution

Financial data has several unique characteristics and complexities that distinguish it from other types of data, necessitating specific approaches for modeling marginal distributions and analysis. Key characteristics include non-normality, where returns exhibit fat tails and skewness; volatility clustering, where periods of high volatility are followed by more high volatility; autocorrelation in volatility measures; leverage effects, where negative returns increase future volatility more than positive returns; and mean reversion, where prices tend to revert to a long-term average (Yu et al., 2024a, 2024b). These features require specialized modeling techniques to accurately capture the data's behavior.

To model these features accurately, it is essential to choose distributions that can handle these properties. The t-distribution captures heavy tails, accommodating extreme values that occur more frequently than in a normal distribution (He et al., 2024). The generalized error distribution (GED) is flexible in handling both heavy tails and skewness effectively (Cerqueti et al., 2019). Skewed distributions, like the skew-normal or skew-t, are used to model asymmetric behavior in financial returns (Wei et al., 2021). Selecting appropriate distributions ensures that the marginal properties of financial returns are accurately represented, which is crucial for any further analysis or modeling.

Financial time series often show volatility clustering, where high volatility periods follow high volatility and low volatility periods follow low volatility. To model this behavior, time-varying volatility models are necessary. GARCH models are standard in financial econometrics for modeling and forecasting volatility (Karimi et al., 2023). GARCH models can be extended to incorporate asymmetries, such as in EGARCH or GJR-GARCH, to account for leverage effects where negative returns impact future volatility more than positive returns (Xiao et al., 2023). Financial data often exhibits non-linear dependencies, especially in volatility measures. To capture these non-linearities, GARCH extensions, such as TGARCH or QGARCH, are designed to capture non-linear volatility patterns (Luan et al., 2022). Regime-switching models allow for different statistical properties in different market conditions, such as high volatility versus low volatility regimes, making them suitable for capturing sudden changes in market behavior (Lee & Lee, 2022; Segnon et al., 2024). To capture both the linear structure in mean returns and the time-varying volatility, combining ARMA with GARCH models is effective. ARMA-GARCH models combine the strengths of ARMA, which captures the autocorrelation in returns, with GARCH, which captures the clustering in volatility (Ly et al., 2022; Yao & Li, 2023; Yu et al., 2024a). This comprehensive approach is powerful for financial time series analysis, providing a robust framework for modeling both mean and variance dynamics.

4. Empirical Applications in Finance

The overlay visualization in Figure 1 illustrates the evolving landscape of literature on copula models in finance. This visualization highlights the frequency and interconnectedness of various key terms associated with copula models, providing insights into the research trends and focal areas over time. Complementing this visualization, Table 2 presents the most prominent keywords in this literature, indicating the areas of highest research activity and interest. Due to their capability to capture intricate dependencies among financial variables, copula models have become increasingly prominent in the financial sector. These models are essential for risk management, portfolio optimization, and derivatives pricing. By accurately modeling dependence structures and tail dependencies, copula models enhance risk assessments, particularly during extreme market conditions. The recent emphasis on volatility and market impact highlights the necessity of understanding market behaviors in the context of global economic uncertainties. Copula models enable more precise calculations of portfolio risk and return, improve the pricing of multi-asset derivatives, and provide superior risk management strategies for commodity markets such as crude oil and gold. Thus, copula models are critical tools that drive progress in financial analytics and risk management.

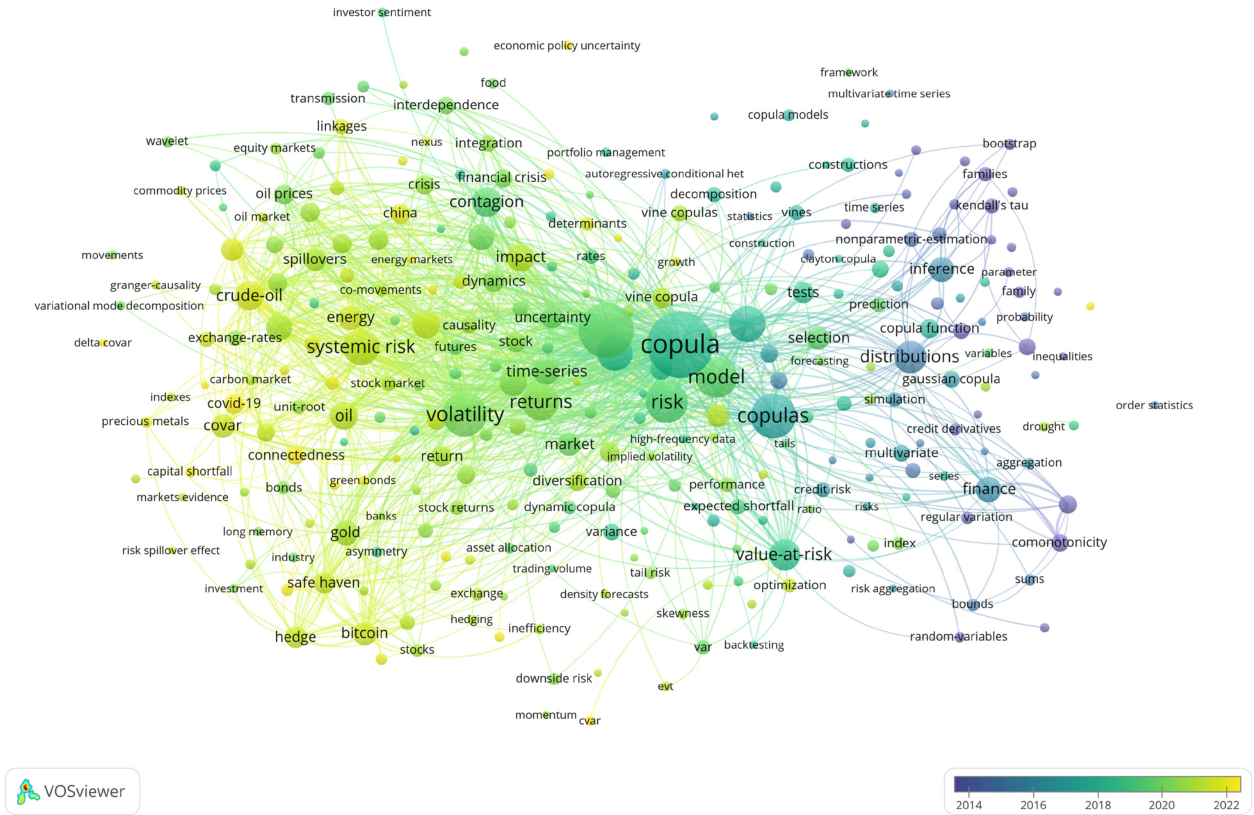

Figure 1. Overlay visualization of copula applications in finance.
Note: 1020 documents are extracted from the Web of Science, and keyword co-occurrence is analyzed using VOSviewer.

Table 2. Most common keywords in copula models in finance (sort by occurrences).

Label	Links	Total link strength	Occurrences	Year
copula	270	1545	292	2018
dependence	250	1270	202	2019
copulas	211	649	134	2017
volatility	212	975	131	2020
risk	198	664	119	2019
model	201	683	115	2019
returns	194	678	92	2020
systemic risk	175	726	89	2021
models	192	517	87	2018
tail dependence	182	509	84	2018
distributions	126	308	72	2015
value-at-risk	140	404	66	2019
contagion	162	445	60	2019
crude-oil	140	546	59	2021
oil	143	414	53	2021
impact	144	399	52	2021
prices	151	406	52	2021
markets	145	384	52	2020
dependence structure	142	323	47	2020
stock markets	133	378	46	2021
time-series	152	346	45	2020

energy	119	312	43	2021
gold	127	391	43	2021

Note: See note in Figure 1.

4.1. Risk Management

Figure 2 presents a keyword analysis of copula applications in risk management. In risk management, copulas have been used to assess the probability of extreme co-movements among assets, which is crucial for calculating Value at Risk (VaR) and Conditional Value at Risk (CVaR). Copulas help in identifying the right tail dependencies, providing a more accurate risk assessment than traditional Gaussian models. "Copula" stands out significantly with high weights and occurrences, reflecting its critical role in financial modeling. Specifically, in the context of risk management, copulas are instrumental for modeling and analyzing dependencies between different risk factors. This allows financial institutions to better understand and manage the joint risks of multiple assets or portfolios. Copulas help in capturing the tail dependencies, which are crucial during extreme market movements where traditional correlation measures may fail. For instance, during financial crises, assets that appear uncorrelated in normal conditions may exhibit strong dependencies. By using copulas, risk managers can more accurately assess and mitigate the impact of such extreme events. The data indicates that copula and related terms like "dependence" and "risk management" have high scores and recent publication years, highlighting the ongoing research and importance of these models in contemporary risk management strategies. This underscores the critical role of copulas in enhancing the robustness and reliability of risk assessment frameworks in finance.

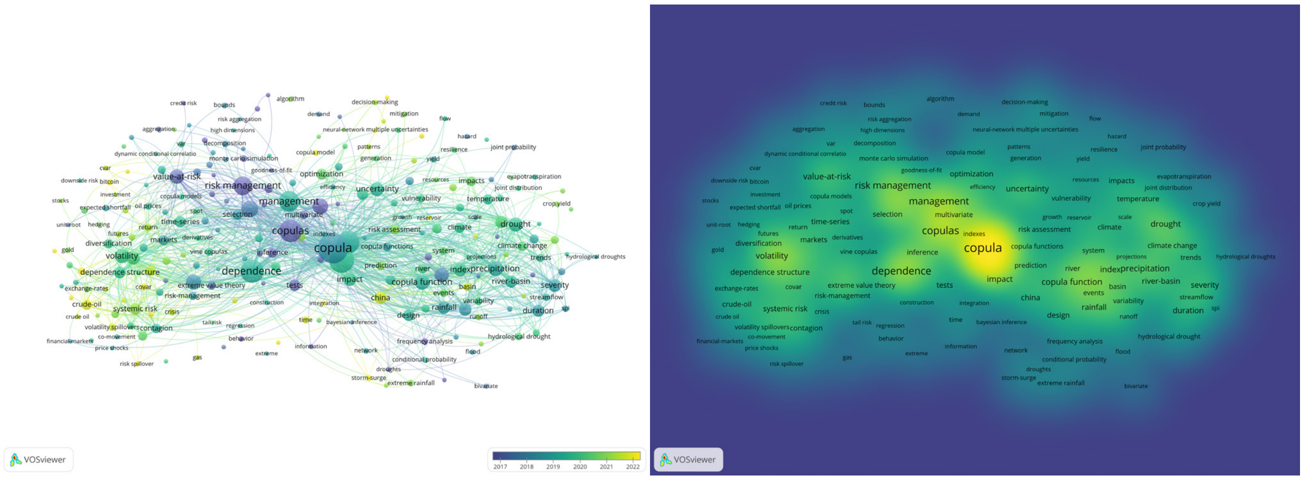

Figure 2. Overlay (left) and density (right) visualizations regarding copula application in risk management.
Note: 715 documents are extracted from the Web of Science, and keyword co-occurrence is analyzed using VOSviewer.

4.2. Portfolio Optimization

Figure 3 presents a keyword analysis of copula applications in portfolio optimization. One key study demonstrated how copula models can be employed to optimize portfolios by accurately modeling the dependence structure between asset returns, thus allowing for a more effective diversification of risk. This approach is particularly useful in tailoring portfolios that are resilient to extreme market movements. The table illustrates the relevance of various financial terms, including "copula" and "portfolio optimization." "Copula" stands out with high weights and occurrences, underscoring its importance in financial modeling. In the context of portfolio optimization, copulas are vital for modeling dependencies between asset returns, which is crucial for accurate risk assessment and optimal asset allocation. Traditional portfolio optimization methods often rely on the assumption that asset returns follow a normal distribution and are linearly correlated. However, these assumptions can be inadequate, especially during market stress when assets exhibit non-linear dependencies. Copula addresses this limitation by providing a more flexible and accurate way to model the joint distribution of asset returns. The high scores and recent average years for both "copula" and "portfolio optimization" highlight the increasing focus on using advanced statistical techniques in finance. By incorporating copulas into portfolio optimization, financial analysts can better capture the true dependency structure among assets, leading to more robust and resilient portfolios. This enhances the ability to manage and mitigate risks, ultimately contributing to improved financial performance and stability.

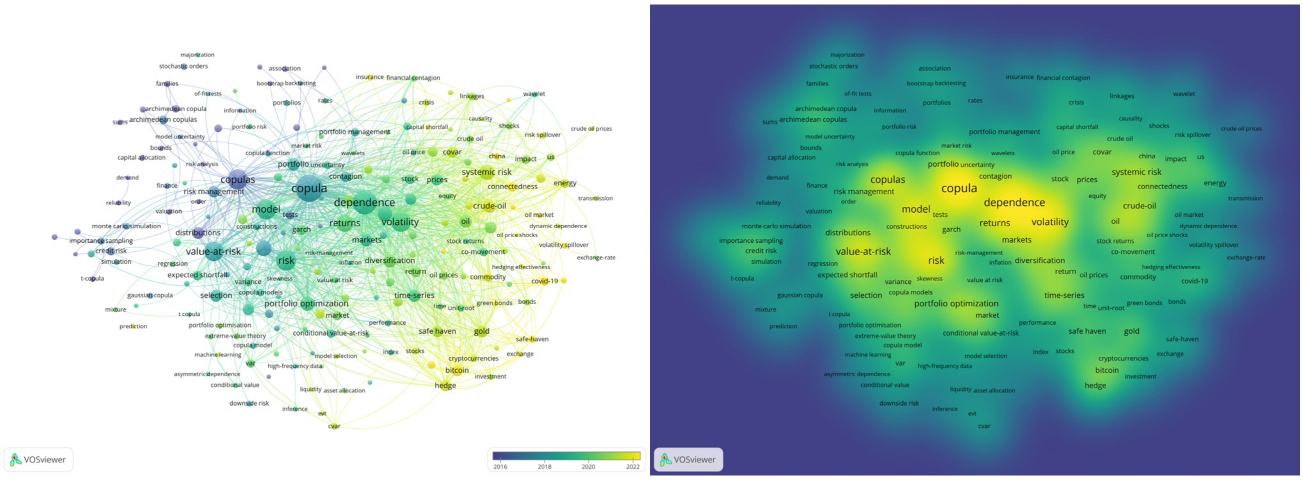

Figure 3. Overlay (left) and density (right) visualizations regarding copula application in portfolio optimization. Note: 654 documents are extracted from the Web of Science, and keyword co-occurrence is analyzed using VOSviewer.

4.3. Derivative Pricing

Figure 4 presents a keyword analysis of copula application in derivative pricing. Copulas are also applied in the pricing of derivatives, especially those involving multiple underlying assets where the joint movements influence the payoff. Copulas enable the accurate modeling of the dependencies between these assets, leading to more reliable pricing models. Focusing on "copula" in the context of derivative pricing, we observe it has a significant weight and occurrence, reflecting its relevance in financial modeling. Copulas are crucial in finance for modeling and understanding the dependencies between different financial instruments or variables. In derivative pricing, they allow for more accurate risk assessment and pricing by capturing the joint distribution of asset returns. This is particularly useful for complex derivatives where the assumption of normal distribution of returns is inadequate. The high scores and recent average years for "copula" and related terms like "copulas" and "dependence" suggest ongoing research and development in this area, underscoring its importance in modern financial engineering and risk management strategies. These models enhance the precision of pricing derivatives by providing a more nuanced view of market dependencies.

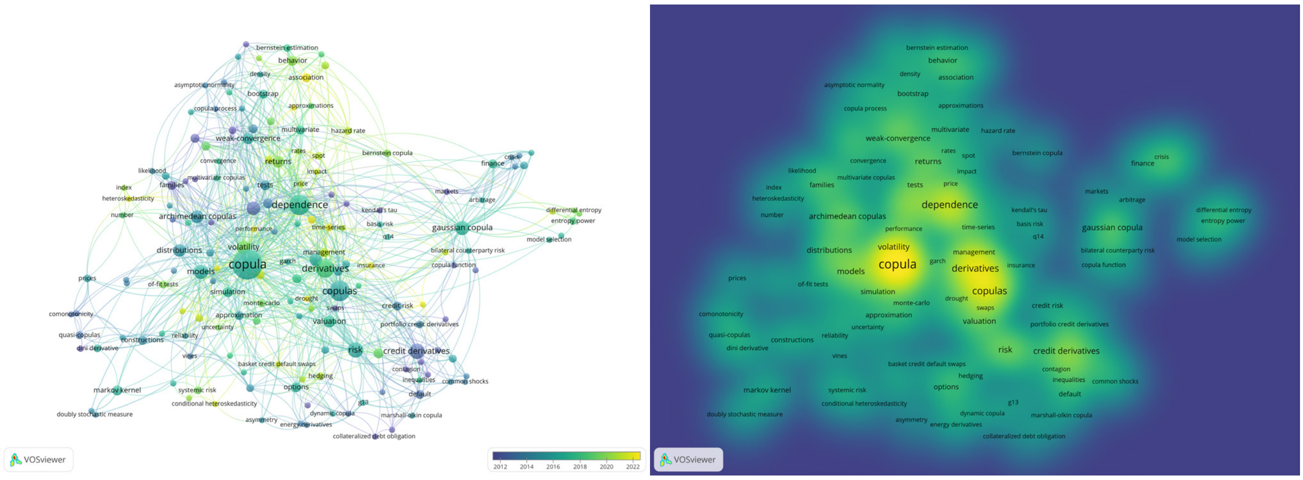

Figure 4. Overlay (left) and density (right) visualizations regarding copula application in derivative pricing.
Note: 163 documents are extracted from the Web of Science, and keyword co-occurrence is analyzed using VOSviewer.

5. Advanced Copula Modeling in Finance

5.1. High Dimension Modeling

Traditional copula models often struggle with capturing complex dependency structures, especially in high-dimensional data. Vine copulas, or pair-copula constructions, offer a flexible solution by decomposing high-dimensional distributions into a series of bivariate copulas. This approach simplifies the modeling process while capturing intricate interactions within larger datasets. Introduced by Joe (1996) and further developed by Bedford & Cooke (2001, 2002), vine copulas can be expressed in terms of distribution functions and densities. They provide a general framework for various configurations, enhancing the ability to model complex dependencies.

There are three primary types of vine copulas: R-vines, C-vines, and D-vines (Nagler et al., 2023). R-vines, or Regular vines, are the most general form and allow for a highly flexible arrangement of variables and dependencies, making them suitable for capturing complex interactions in high-dimensional data. They use a tree structure to define the pairwise copulas and their dependencies. C-vines, or Canonical vines, have a hierarchical structure where one variable is chosen as the root, and all pairwise dependencies are conditioned on this root. This central variable simplifies the modeling process by focusing on its relationships with all other variables and is particularly useful when there is a natural choice for a central variable that influences the others. D-vines, or Drawable vines, model dependencies sequentially, conditioning only on preceding variables (Aas et al., 2009). This type of vine copula allows for a more flexible and straightforward representation of dependencies, making it easier to interpret and implement, and is useful when the data naturally follows a sequential order, such as time series data (Czado, 2019).

5.2. Dynamic Copula

As the copula density function can be dynamic, the copula dependence parameters are allowed to be time-varying while the copula function remains unchanged (Hafner & Manner, 2012; Nguyen et al., 2019; Patton, 2006). An important feature of any dynamic model is to specify how the parameters evolve through time. Such models can be classified into two classes: observation-driven and parameter-driven specifications. The parameter-driven specifications, such as stochastic copula models allow the varying parameters to evolve as a latent time series process with idiosyncratic innovations (Hafner & Manner, 2012). The observation-driven specifications, such as ARCH-type models for volatility and related models for copulas model the varying parameters as some function of lagged dependent variables as well as contemporaneous and lagged exogenous variables (Creal et al., 2013; Patton, 2006). In this approach, the parameters evolve randomly over time, but they are perfectly predictable one step ahead given past information. The likelihood function for such models is also available in closed form. Another advantage of the latter approach over the former is that it avoids the need to ''integrate out'' the innovation terms driving the latent time series processes. However, within the class of observation-driven specifications, the choice of an appropriate function of lagged dependent variables is to be made. For models of the conditional variance, the lagged squared residual (the ARCH-family of models) comes as an obvious choice, but for models with parameters that lack an obvious interpretation, the choice is less clear. To overcome this problem, we follow Creal et al. (2013) and Harvey (2013), and allow the time-varying parameter to follow the generalized autoregressive score (GAS) process. The process adopts the score vector of the predictive model density to update the time-varying parameters. This choice is motivated by the fact that the GAS model belongs to a class of observation-driven models with a similar degree of generality as obtained for non-linear, non-Gaussian state-space models. By relying on the density structure to update the time-varying parameters, GAS models take into account full information in the data distribution. Koopman et al. (2016) provide empirical evidence that the GAS-updated process outperforms other observation-driven processes in terms of predictive accuracy. In the bivariate context, several copula functions allow for a flexible dependence (Joe, 2014). Besides, elliptical and Archimedean copula families are most commonly used in finance due to parsimonious specification and their ability to capture tail dependence.

6. Software Tools

The availability of free software has significantly contributed to the success of vine copula models. Due to the complexity of the algorithms involved in inference and simulation, their implementation demands considerable effort and expertise. This burden was alleviated early on for applied researchers by the R package "CDVine" developed by Brechmann & Czado (2013), which was first released in May 2011. However, this package was initially limited to C- and D-vines, reflecting the state of research at the time.

Currently, the most popular package is "VineCopula" by Nagler et al. (2023), which is the successor to "CDVine". This package supports arbitrary vine structures and includes additional copula families, offering extensive functionality for modeling dependence with both bivariate and vine copulas. Its capabilities encompass statistical functions (such as densities, distributions, and simulation), inference algorithms (for parameter estimation and model selection), and tools for exploratory data analysis and visualization, covering most of the content in this book. Another recent alternative is the "vinecopulib" project (www.vinecopulib.org), which features an efficient C++ implementation of the key features of VineCopula, making it particularly useful for high-dimensional applications. This project also supports mixing parametric and nonparametric pair copulas and provides interfaces for both R and Python. Additionally, there is a MATLAB toolbox for vine copulas, supported by an associated C++ library (Kurz, 2015). Moreover, several other specialized R packages related to vine copula models are available. Table 3 lists some of the packages and libraries for copula modeling.

Table 3. Software tools for copula modeling.

Packages/Libraries	Language	Maintainer
CDVine	R	Brechmann & Czado (2013)
CDVineCopulaConditional	R	Bevacqua (2017)
copula	R	Hofert et al. (2023)
gamCopula	R	Vatter & Nagler (2017)
kdevine	R	Nagler (2017a)
pencopulaCond	R	Schellhase (2017a)
penRvine	R	Schellhase (2017b)
rvinecopulib	R	Nagler & Vatter (2023)

VineCopula	R	Nagler et al. (2023)
vinereg	R	Nagler (2017b)
copulae	Python	Bok (2024)
copulalib	Python	Tomer (2022)
copulas	Python	DataCebo (2024)
pycop	Python	Nicolas (2024)
pycopula	Python	Jumelle (2018)
pyvinecopulib	Python	Vatter (2024)
VineCopulas	Python	Claassen (2024)
MATVines	MATLAB	Coblenz (2021)

7. Future Directions and Challenges

In the future, copula models in finance are expected to focus on several key areas: analyzing the impact of China's market and global events, particularly the long-term effects of COVID-19 on financial dependencies; assessing risk structures associated with clean energy investments and green bonds to support sustainable finance; studying the dependencies in the rapidly evolving technology and cryptocurrency markets; and enhancing the understanding of volatility spillover effects and financial risks. These directions, indicated by the recent hot topics in Table 4, will enable copula models to play a more significant role in complex and dynamic financial environments. In addition to exploring hot research topics, there's also a significant focus on methodological refinements. Although traditional parametric copula methods have significant advantages in modeling multivariate distributions and capturing complex dependencies between variables, they also have some drawbacks. For instance, selecting the appropriate copula function and estimating parameters can be complex and sensitive to insufficient or inaccurate data. Additionally, different types of copulas vary in their effectiveness at handling tail dependencies and extreme values. To address these issues, a combination of hybrid copula, machine learning, and artificial intelligence can be utilized (Kwok et al., 2024; Mehta et al., 2023).

Table 4. Recent hot topics in copula models in finance (sort by year).

Label	Links	Total link strength	Occurrences	Year
bernstein copula	11	14	5	2023
capital shortfall	43	59	8	2022
china	98	176	27	2022
clean energy	46	61	6	2022
connectedness	89	172	22	2022
covid-19	98	195	27	2022
cryptocurrency	44	74	10	2022
cvar	40	60	7	2022
delta covar	40	51	5	2022
economic policy uncertainty	22	22	6	2022
empirical-evidence	21	22	5	2022
energy markets	41	49	6	2022
financial risk	33	39	7	2022
gas	35	44	6	2022
green bonds	35	49	7	2022
oil price shocks	46	56	6	2022
quantile dependence	36	43	6	2022
renewable energy	42	56	8	2022
safe-haven	63	113	11	2022
technology	43	57	8	2022
volatility spillover	72	99	13	2022

Note: See note in Figure 1.

7.1. Hybrid Copula

Hybrid copulas present an innovative approach to enhance the modeling of multivariate dependencies by leveraging the strengths of multiple copula functions. The central concept involves utilizing different copulas for various segments of the data or combining them in ways that more effectively capture the diverse characteristics of dependencies (Wang et al., 2022). A significant advantage of hybrid copulas is their capacity to tailor dependency structures to different parts of the distribution. Another approach involves combining multiple copulas with weighted schemes, where weights are determined based on empirical evidence from the data (Bianchi et al., 2023). Additionally, hybrid copulas offer enhanced flexibility in parameterization, allowing for better alignment with the specific characteristics of the data. In contemporary research, there is a growing interest in nonparametric copulas over traditional parametric ones. This shift stems from nonparametric copulas not relying on pre-assumed functional forms for dependency structures. Instead, they employ data-driven techniques for copula estimation (Djaloud & Seck, 2024; Ho et al., 2019; Neumeyer et al., 2019). Among nonparametric copulas, Bernstein copulas are notable as they are constructed using Bernstein polynomials (Hernández-Maldonado et al., 2024). These copulas offer advantages such as smoothness and adaptability. Bernstein copulas are particularly valuable because they can closely approximate any continuous copula, providing a potent tool for modeling intricate dependencies (Bahraoui, 2023; Scheffer & Weiß, 2017). In addition, semiparametric copulas combine both parametric and nonparametric elements in their construction. Typically, the marginal distributions of the random variables are estimated nonparametrically, while the corresponding copula function capturing the dependency structure remains parametric (X. Chen et al., 2024; Cheng et al., 2014).

7.2. ML & AI Integration

ML techniques offer robust solutions for enhancing copula models, particularly in dealing with large and intricate datasets. The amalgamation of ML with copula models can enhance various facets of the modeling process. ML algorithms streamline the preprocessing of extensive datasets by automating tasks like cleansing, normalization, and transformation. Methods such as Principal Component Analysis (PCA) and t-distributed Stochastic Neighbor Embedding (t-SNE) effectively reduce dimensionality and accentuate crucial features (Huang et al., 2022; Tian et al., 2020). Sophisticated ML techniques like convolutional neural networks (CNNs) and recurrent neural networks (RNNs) excel in extracting intricate features from data (Lu & Xu, 2024; Zhao et al., 2021), thereby uncovering latent patterns and dependencies that traditional statistical methods might overlook, enriching inputs for copula models. Additionally, ML algorithms such as genetic algorithms (Bedoui et al., 2023) and particle swarm optimization (Jagtap et al., 2020) efficiently optimize copula parameters, navigating the complex parameter spaces to identify the most fitting parameters.

AI extends the capabilities of machine learning by integrating elements of cognitive computing and advanced algorithms to automate and refine the copula modeling process. AI systems automate the construction of copula models, from data ingestion to model fitting and validation. Automated machine learning (AutoML) platforms streamline this process, enabling swift development and deployment of copula models (T. Chen et al., 2021). AI proficiently manages and preprocesses vast datasets in real-time, identifying anomalies, outliers, and significant patterns that require attention prior to modeling. Natural Language Processing (NLP) can extract pertinent information from unstructured data sources like financial news and reports, providing additional context for modeling (Kesgin & Amasyali, 2024).

8. Conclusions

Copula theory, a mathematical framework from statistics and probability, has been extensively reviewed in this paper for its application in modeling financial asset dependencies. This review covers the theoretical foundations, various modeling techniques, empirical applications, and practical implementation of copula theory in finance. The findings highlight the copula theory's strength in capturing complex, non-linear dependencies and tail behaviors that traditional correlation methods often miss. It is particularly effective in risk management, portfolio optimization, and derivative pricing under extreme market conditions.

The paper discusses the application of vine copulas and dynamic copulas, which provide solutions for high-dimensional and time-varying dependencies, respectively. Vine copulas decompose high-dimensional distributions into simpler bivariate copulas, while dynamic copulas incorporate time-varying parameters to enhance model responsiveness. These advanced models demonstrate the versatility and adaptability of copula theory in addressing various financial modeling challenges. However, the paper also notes the challenges in applying copula theory due to its mathematical complexity and the sensitivity of parameter estimation to data quality. Selecting appropriate copula functions and accurately estimating parameters require significant expertise and careful calibration. The integration of machine learning and artificial intelligence presents promising future directions, potentially enhancing model accuracy and usability by automating complex processes and handling large datasets more efficiently.

Overall, this review underscores the significant contributions of copula theory in providing deeper insights into financial asset dependencies and improving the robustness of financial risk management and analysis. The continued advancement of copula-based models promises further accuracy and reliability in financial decision-making, offering a valuable toolset for practitioners and researchers in the evolving landscape of modern finance.

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